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Crossing symmetry in elliptic solutions of the Yang–Baxter equation and a new L-operator for Belavin's solution

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Abstract. Some algebraic structures in elliptic solutions of the Yang–Baxter equations are investigated. We prove the crossing symmetry in Belavin's model as well as in the $A_{n-1}^{(1)}$ face model, and we construct a new family of L-operators for Belavin's R-matrix as an application.

1. Introduction

Recently much progress has been made in the theory of two-dimensional solvable statistical lattice models. Here we shall investigate some algebraic structures in elliptic solutions of the Yang–Baxter equations (YBE). In particular, we demonstrate the crossing symmetry in Belavin's model [1] as well as in Jimbo *et al* $A_{n-1}^{(1)}$ face model [2], and we construct a new family of L-operators for Belavin's model as an application.

Bazhanov and Stroganov [3] showed that the chiral Potts model, which is a solution of the YBE or the star-triangle relation whose spectral parameter lies in a high genus algebraic curve [4, 5], is a 'descendent' of the 6-vertex model which is nothing but the R-matrix associated to $U_q(\widehat{\mathrm{sl}}_n)$. That is, they derived the chiral Potts model as the intertwiner of cyclic L-operators or, equivalently, the intertwiner of the two-fold tensor of cyclic representations of $U_q(\widehat{\mathrm{sl}}_n)$. Motivated by their result, in our previous paper [6] we showed that Kashiwara–Miwa's elliptic solution (the so-called broken $\mathbb{Z}_N$ symmetric solution [7]) is a descendent of Baxter's 8-vertex model [8], i.e. if we take Sklyanin's cyclic L-operator for the 8-vertex model we obtain Kashiwara–Miwa's model as the intertwiner for the L-operators. Together with this derivation, in [9] we further succeeded in relating the crossing symmetry of Kashiwara–Miwa's model with a certain duality property of the L-operators.

To generalize this story for the n-state elliptic model of Belavin, one immediately needs a cyclic L-operator for the model and its construction is one of our objectives here. We are inspired by an idea in Bazhanov *et al* [10]. They considered the $U_q(\widehat{\mathrm{sl}}_n)$ generalization in [3] by means of 'intertwining vectors' or 'factorized L-operators [11]'. Intertwining vectors were originally introduced in [12] to introduce face models via vertex models. Hence by definition they relate a vertex model and a certain face model, and using this relationship [10] observed that a simple combination of intertwining vectors provides an L-operator. Intertwining vectors between Belavin's model and the $A_{n-1}^{(1)}$ face model are given in [13] and they are, so to speak, 'outgoing' intertwining vectors. What we need to construct L-operators are their 'dual' or 'incoming' intertwining vectors and our method for constructing them is as follows. We first observe the crossing symmetry of the models, which is nothing but the incoming/outgoing duality (sections 3 and 4), and then we obtain the incoming intertwining vectors by fusing [2, 14, 15] the original intertwining vectors (section 5). The resulting L-operators (section 6) act on $\mathbb{C}^n \otimes$ (space of functions on the weight space h^* of sl_n), and of letting the deformation parameter $q = e^{\hbar}$ of Belavin's model be a root of unity invariant subspaces arise and we can obtain the desired cyclic L-operators. In addition to this cyclic one we can also find other invariant subspaces [16] so that we can generalize the analogue of Sklyanin's series of L-operators [17] for Baxter's 8-vertex model to the Belavin model.

As is well known, up to a certain transformation the trigonometric limit of Belavin's model gives the R-matrix of $U_q(\widehat{\mathrm{sl}}_n)$ in the vector representation [18]. What we have observed here should be regarded as part of the theory of the 'elliptic' version of quantum groups [14, 19, 20] and we have formulated the model keeping this in mind. We hope that this paper will give some insight into the nature of the theory.

2. Review

2.1. Belavin's vertex model [1]

For $n > 1$ let $\mathbb{C}^n = \oplus_{k \in \mathbb{Z} \bmod n\mathbb{Z}} \mathbb{C}e^k$ and let $g, h \in \mathrm{GL}(\mathbb{C}^n)$ to be $ge^k := e^k \exp 2\pi i k/n$, $he^k := e^{k+1}$ so that $gh = hg \exp 2\pi i/n$. Let $\hbar, \tau \in \mathbb{C}, \hbar \neq 0, \mathrm{Im}\,\tau > 0$. Belavin's R-matrix is characterized as the unique solution of the following five conditions.

(i) $R(u)$ is a holomorphic $\mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$-valued function in u

(ii) $R(u) = (x \otimes x)R(u)(x \otimes x)^{-1}$ for $x = g, h$

(iii) $R(u+1) = (g \otimes 1)^{-1} R(u)(g \otimes 1) \times (-1)$ (1)

(iv) $R(u+\tau) = (h \otimes 1)R(u)(h \otimes 1)^{-1} \times (-\exp 2\pi i(u + (\hbar/n) + \frac{1}{2}\tau))^{-1}$

(v) $R(0) = P : x \otimes y \mapsto y \otimes x$.

We also have the following formula for $R(u)$ [21]:

$$R(u)e^i \otimes e^j = \sum_{i',j'} e^{i'} \otimes e^{j'} R(u)^{ij}_{i'j'}$$

$$R(u)^{ij}_{i'j'} = \delta_{i+j,i'+j' \bmod n} \frac{\theta^{(i'-j')}(u+\hbar)}{\theta^{(i'-i)}(\hbar)\theta^{(i-j')}(u)} \frac{\prod_{k=0}^{n-1} \theta^{(k)}(u)}{\prod_{k=1}^{n-1} \theta^{(k)}(0)}. \tag{2}$$

Here

$$\theta_{m,l}(u,\tau) := \sum_{\mu \in m + l\mathbb{Z}} \exp 2\pi i \left(\mu u + \frac{\mu^2}{2l}\tau \right)$$

and

$$\theta^{(j)}(u) := \theta_{1/2 - j/n, 1}(u + \tfrac{1}{2}, n\tau).$$

Then the YBE of the vertex type

$$R^{23}(u_2 - u_3)R^{13}(u_1 - u_3)R^{12}(u_1 - u_2) = R^{12}(u_1 - u_2)R^{13}(u_1 - u_3)R^{23}(u_2 - u_3) \tag{3}$$

holds on $V_1 \otimes V_2 \otimes V_3$, where V_i are copies of $\mathbb{C}^n$ and R^{ij} acts on ith and jth spaces.

For the latter purpose we will reformulate this solution as follows. For each $u \in \mathbb{C}$ let $V(\square_u)$ be the copy of $\mathbb{C}^n$ and write the R-matrix $R(u - v)$ acting on $V(\square_u) \otimes V(\square_v)$ as

$R^{\Box_u,\Box_v}$. We also put $\check{R}^{\Box_u,\Box_v} := PR^{\Box_u,\Box_v}$, where P is the permutation $V(\Box_u)\otimes V(\Box_v)\to V(\Box_v)\otimes V(\Box_u)$. Then YBE (3) reads as follows

$$(\check{R}^{\Box_v,\Box_w}\otimes 1)(1\otimes\check{R}^{\Box_u,\Box_w})(\check{R}^{\Box_u,\Box_v}\otimes 1)=(1\otimes\check{R}^{\Box_u,\Box_v})(1\otimes\check{R}^{\Box_u,\Box_w})(\check{R}^{\Box_v,\Box_w}\otimes 1)$$
$$: V(\Box_u)\otimes V(\Box_v)\otimes V(\Box_w)\to V(\Box_w)\otimes V(\Box_v)\otimes V(\Box_u). \tag{4}$$

Remark 1. The notation $\Box$ stands for the Young diagram consisting of one box. If we consider the 'algebra of L-operators' for Belavin's R-matrix then the notation $\Box_u$ can be justified as its 'vector representation with the spectral parameter u'.

2.2. $A^{(1)}_{n-1}$ face model [13]

Let $\epsilon_i (i=1,\ldots,n)$ be the orthonomal basis of an n-dimensional vector space with the inner product $(,)$ and put $h^* :=$ $\mathbb{C}$-Span of $\{\epsilon_i-\epsilon_{i+1}(i=1,\ldots,n-1)\}$ so that we can identify h^* and the weight space of the complex Lie algebra sl_n in a usual way. Let $\bar{\ }:\mathbb{C}^n\to h^*$ be the orthogonal projection. Then the Boltzmann weight of the $A^{(1)}_{n-1}$ face model corresponding to the vector representation $\Box$ is given by the following.

$$\check{W}\begin{bmatrix} & \lambda+\bar{\epsilon}_i & \\ \lambda & u & \lambda+2\bar{\epsilon}_i \\ & \lambda+\bar{\epsilon}_i & \end{bmatrix} := \frac{h(u+\hbar)}{h(\hbar)}$$

$$\check{W}\begin{bmatrix} & \lambda+\bar{\epsilon}_i & \\ \lambda & u & \lambda+\bar{\epsilon}_i+\bar{\epsilon}_j \\ & \lambda+\bar{\epsilon}_i & \end{bmatrix} := \frac{h(-u+\hbar\lambda_{ij})}{h(\hbar\lambda_{ij})}$$

$$\check{W}\begin{bmatrix} & \lambda+\bar{\epsilon}_i & \\ \lambda & u & \lambda+\bar{\epsilon}_i+\bar{\epsilon}_j \\ & \lambda+\bar{\epsilon}_j & \end{bmatrix} := \frac{h(u)}{h(\hbar)}\frac{h(\hbar+\hbar\lambda_{ij})}{h(\hbar\lambda_{ij})}$$

and, for the other configuration of λ,μ,μ' and ν,

$$\check{W}\begin{bmatrix} & \mu & \\ \lambda & u & \nu \\ & \mu' & \end{bmatrix} := 0$$

where $\lambda_{ij} := (\lambda+\rho,\bar{\epsilon}_i-\bar{\epsilon}_j)$, $\rho := \sum_{j=1}^{n}(n-j)\bar{\epsilon}_j$ is the half sum of positive roots and

$$\begin{aligned} h(u) &:= \theta_{1/2,1}(u+\tfrac{1}{2},\tau) \\ &= \mathrm{i}q^{1/8}(z^{1/2}-z^{-1/2})\prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m) \end{aligned} \tag{5}$$

where $q := \exp 2\pi\mathrm{i}\tau$, $z := \exp 2\pi\mathrm{i}u$. The function h satisfies

$$h(u+1) = -h(u) \qquad h(u+\tau) = -h(u)\exp 2\pi\mathrm{i}[-u-\tau/2] \tag{6}$$

and $h(u)=0$ for $u\in\mathbb{Z}+\tau\mathbb{Z}$.

To formulate the weight $\check{W}$ as a linear operator or a 'face operator (an elementary transfer matrix)', the following vector space is in order.

$$\mathcal{P}^{\mu}_{\lambda\Box} :\cong \begin{cases} \mathbb{C} : \mu=\lambda+\bar{\epsilon}_i & \text{for some } i \\ 0 : & \text{otherwise.} \end{cases}$$

We denote by e_λ^μ the basis of the one-dimensional space $\mathcal{P}_{\lambda\Box}^\mu$ when $\mu = \lambda + \bar{\epsilon}_i$ for some i, and otherwise we set $e_\lambda^\mu = 0$. For each $u \in \mathbb{C}$ we consider the copy $\mathcal{P}_{\lambda\Box_u}^\mu$ of $\mathcal{P}_{\lambda\Box}^\mu$ and define

$$\mathcal{P}_{\lambda\Box_{u_1}\cdots\Box_{u_k}}^\nu := \sum_{\mu_1,\ldots,\mu_{k-1}} \mathcal{P}_{\lambda\Box_{u_1}}^{\mu_1} \otimes \mathcal{P}_{\mu_1\Box_{u_2}}^{\mu_2} \otimes \cdots \otimes \mathcal{P}_{\mu_{k-1}\Box_{u_{k-1}}}^\nu$$

$$\mathcal{P}_{\Box_{u_1}\cdots\Box_{u_k}} := \oplus_{\lambda,\nu}\mathcal{P}_{\lambda\Box_{u_1}\cdots\Box_{u_k}}^\nu .$$

This is the space of 'admissible paths' in [2]. For $e_\lambda^\mu \in \mathcal{P}_{\lambda\Box_\mu}^\mu$ and $e_\mu^\nu \in \mathcal{P}_{\mu\Box_\nu}^\nu$ we put

$$\check{W}^{\Box_u,\Box_v}(e_\lambda^\mu \otimes e_\mu^\nu) := \sum_{\mu'} e_\lambda^{\mu'} \otimes e_{\mu'}^\nu \check{W}\begin{bmatrix} & \mu & \\ \lambda & u-v & \nu \\ & \mu' & \end{bmatrix}$$

thereby defining the face operator

$$\check{W}^{\Box_u,\Box_v} : \mathcal{P}_{\Box_u\Box_v} \to \mathcal{P}_{\Box_v\Box_u} \qquad \mathcal{P}_{\lambda\Box_u\Box_v}^\nu \to \mathcal{P}_{\lambda\Box_v\Box_u}^\nu .$$

With these definitions the YBE of face type reads as follows. As operators $\mathcal{P}_{\nu\Box_u\Box_v\Box_w}^\lambda \to \mathcal{P}_{\nu\Box_w\Box_v\Box_u}^\lambda$ we have

$$(1 \otimes \check{W}^{\Box_u,\Box_v})(\check{W}^{\Box_u,\Box_w} \otimes 1)(1 \otimes \check{W}^{\Box_v,\Box_w}) = (\check{W}^{\Box_v,\Box_w} \otimes 1)(1 \otimes \check{W}^{\Box_u,\Box_w})(\check{W}^{\Box_u,\Box_v} \otimes 1). \tag{7}$$

2.3. Intertwining vectors [13]

Put

$$(\phi_{\lambda\Box_u}^\mu)_j := \begin{cases} \theta^{(j)}(u - n\hbar(\lambda+\rho,\bar{\epsilon}_k)) : \mu - \lambda = \bar{\epsilon}_k & \text{for some } k \\ 0 : & \text{otherwise} \end{cases}$$

and define the linear map

$$\phi_{\lambda\Box_u}^\mu : \mathcal{P}_{\lambda\Box_u}^\mu \to V(\Box_u)$$

by

$$\phi_{\lambda\Box_u}^\mu e_\lambda^\mu := \sum_j e^j (\phi_{\lambda\Box_u}^\mu)_j .$$

Then the $\phi_{\lambda\Box}^\mu$ 'intertwine' Belavin's vertex model and the $A_{n-1}^{(1)}$ face model, namely

$$\check{R}^{\Box_u,\Box_v}\phi_{\mu\Box_u}^\lambda \otimes \phi_{\nu\Box_v}^\mu = \sum_{\mu'} \phi_{\mu'\Box_v}^\lambda \otimes \phi_{\nu\Box_u}^{\mu'} \check{W}\begin{bmatrix} & \mu & \\ \lambda & u-v & \nu \\ & \mu' & \end{bmatrix}. \tag{8}$$

This formula is remarkable because of its similarity to the monodoromy property of the n-point function in the q-conformal field theory [22] and between Pasquire's formulation of the face models [23].

The quantity $\{(\phi_{\mu\Box_u}^\lambda)_j\}_{j=1}^n$ regarded as an n-vector is called the *intertwining vector*.

3. Crossing symmetry in the vertex models

3.1. Fusion procedure [14]

Let

$$\check{R}^{\Box_{u_1}\otimes\Box_{u_2}\otimes\cdots\otimes\Box_{u_l},\Box_v} := (\check{R}^{\Box_{u_1},\Box_v})^{1,2}(\check{R}^{\Box_{u_2},\Box_v})^{2,3}\cdots(\check{R}^{\Box_{u_l},\Box_v})^{l,l+1}$$

$$: V(\Box_{u_1})\otimes\cdots\otimes V(\Box_{u_k})\otimes V(\Box_v)\to V(\Box_v)\otimes V(\Box_{u_1})\otimes\cdots\otimes V(\Box_{u_k})$$

$$\check{R}^{\Box_{u_1}\otimes\cdots\otimes\Box_{u_k},\Box_{v_1}\otimes\cdots\otimes\Box_{v_l}} :$$

$$= (\check{R}^{\Box_{u_1}\otimes\cdots\otimes\Box_{u_k},\Box_{v_1}})^{k\cdots k+l-1;k+l}\cdots$$

$$\times(\check{R}^{\Box_{u_1}\otimes\cdots\otimes\Box_{u_k},\Box_{v_2}})^{2\cdots k+1;k+2}(\check{R}^{\Box_{u_1}\otimes\cdots\otimes\Box_{u_k},\Box_{v_l}})^{1\cdots k;k+l}$$

$$: V(\Box_{u_1})\otimes\cdots\otimes V(\Box_{u_k})\otimes V(\Box_{v_1})\otimes\cdots\otimes V(\Box_{v_l})$$

$$\to V(\Box_{v_1})\otimes\cdots\otimes V(\Box_{v_l})\otimes V(\Box_{u_1})\otimes\cdots\otimes V(\Box_{u_k}).$$

For $k=1,\ldots,n$ let 1^k be the Young diagram of vertical k boxes ($1^1=\Box$; in this paper we will treat these special diagrams for simplicity). Then the fusion operator by Cherednik [14] associated with 1^k is given by

$$\pi_{1^k} : V(\Box_u)\otimes\cdots\otimes V(\Box_{u+(k-1)\hbar})\to V(\Box_{u+(k-1)\hbar})\otimes\cdots\otimes V(\Box_u)$$

$$:=(\check{R}^{\Box_{u_1},\Box_{u_2}})^{k-1;k}\cdots(\check{R}^{\Box_{u_1}\otimes\Box_{u_2}\otimes\cdots\otimes\Box_{u_{k-2}},\Box_{u_{k-1}}})^{2\cdots k-1;k}$$

$$\times(\check{R}^{\Box_{u_1}\otimes\Box_{u_2}\otimes\cdots\otimes\Box_{u_{k-1}},\Box_{u_k}})^{1\cdots k-1;k} \tag{9}$$

where the spectral parameters are specialized as

$$(u_1,\ldots,u_k)=(u,u+\hbar,\ldots,u+(k-1)\hbar) \tag{10}$$

so that the rank of the operator π degenerates. By virtue of YBE (5) the factors in (9) can be arranged in various ways by 'braid manipulation' and this is the key remark in deriving the formula in what follows. We denote the image of π_{1^k} in $V(\Box_{u+(k-1)\hbar})\otimes\cdots\otimes V(\Box_u)$ as $V(1^k_u)$ and then it turns out that $V(1^k_u)=\wedge^k(\mathbb{C}^n)$ for the generic value of $\hbar$. Put

$$\check{R}^{K,L} := \check{R}^{\Box_{u+(k-1)\hbar}\otimes\cdots\otimes\Box_u,\Box_{v+(l-1)\hbar}\otimes\cdots\otimes\Box_v}|_{V(K)\otimes V(L)}$$

where

$$K=1^k_u \qquad L=1^l_v$$

is the shorthand notation. Then the YBE for $\check{R}^{\Box_u,\Box_v}$ (5) guarantees that this 'fused' operator preserves the image of π

$$\check{R}^{K,L} : V(K)\otimes V(L)\to V(L)\otimes V(K)$$

as well as the YBE: for $K=1^k_u, L=1^l_v, M=1^m_w$ we have

$$(\check{R}^{L,M}\otimes 1)(1\otimes\check{R}^{K,M})(\check{R}^{K,L}\otimes 1)=(1\otimes\check{R}^{K,L})(\check{R}^{K,M}\otimes 1)(1\otimes\check{R}^{L,M})$$

$$: V(K)\otimes V(L)\otimes V(M)\to V(M)\otimes V(L)\otimes V(K). \tag{11}$$

3.2. Crossing symmetry

Let us denote the special diagram 1^n as *top* and put

$$(1^m_w)^* := 1^{n-m}_{w+m\hbar}. \tag{12}$$

Then since $\pi_{\text{top}} = \check{R}^{K,K^*}(\pi_K \otimes \pi_{K^*})$ for each $K = 1^k_u$, we can define a pairing

$$\langle , \rangle : V(K) \otimes V(K^*) \to V(\text{top}_u) \to \mathbb{C} \tag{13}$$

as the composition of $\check{R}^{K,K^*}$ and the identification map $|\text{top}_u\rangle \mapsto 1$, where $|\text{top}_u\rangle$ is a fixed basis of the one-dimensional space $V(\text{top}_u)$.

For generic $\hbar$ this pairing turns out to be non-degenarate so that we can and do identify $V(K)^*$ and $V(K^*)$, where $V(K)^*$ stands for the dual space of $V(K)$. Fix $K = 1^k_u$ and $L = 1^l_v$. We take a basis $\{e^I\}_I$ in $V(K)$ and its dual basis (with respect to $\langle , \rangle$) $\{e^I_*\}_I$ in $V(K^*)$, and do the same for L. We define the matrix elements of $\check{R}$ by

$$\check{R}^{K,L} e^I \otimes e^J = \sum_{I',J'} e^{J'} \otimes e^{I'} (\check{R}^{K,L})^{IJ}_{J'I'} \qquad (e^I \in V(K), e^J \in V(L))$$

$$\check{R}^{K,L^*} e^I \otimes e^J_* = \sum_{I',J'} e^{J'}_* \otimes e^{I'} (\check{R}^{K,L^*})^{IJ}_{J'I'} \qquad (e^I \in V(K), e^J_* \in V(L^*))$$

etc.

Proposition 1. Let $K = 1^k_u, L = 1^l_v$ and top $= 1^n$. Then under notation (12) we have the following.

(i) There is a scalar $f(K, L)$ which is non-zero for generic u, v such that

$$\check{R}^{L,K} \check{R}^{K,L} = f(K, L) \cdot \text{id}_{V(K)\otimes V(L)}. \tag{14}$$

(ii) We have

$$\check{R}^{K,\text{top}_v} = g(K, \text{top}_v) \cdot P \qquad \check{R}^{\text{top}_u,L} = g(\text{top}_u, L) \cdot P \tag{15}$$

where $P : V(K) \otimes V(\text{top}_v) \to V(\text{top}_v) \otimes V(K)$ is the permutation of the components and $g(K, \text{top}_v)$, $g(\text{top}_u, L)$ are scalars which are non-zero for generic u, v.

(iii) The following crossing symmetry holds:

$$(\check{R}^{K,L})^{IJ}_{J'I'} = (\check{R}^{L,K^*})^{JI'}_{IJ'} \frac{f(K, L)}{g(L, \text{top}_u)} \tag{16}$$

$$= (\check{R}^{L,K^*})^{JI'}_{IJ'} \frac{g(\text{top}_u, L)}{f(L, K^*)}. \tag{17}$$

Proof. (i) follows from the first inversion formula $\check{R}^{\Box_v,\Box_u} \check{R}^{\Box_u,\Box_v} =$ scalar for Belavin's original R-matrix. To show (ii), note that the operator π_K commutes with the k-fold tensor product representation of the Heisenberg group $\langle g, h\rangle$ (1). Since $\dim V(\text{top}_u) = 1$, the representation restricted on $V(\text{top}_u)$ is only by scalar multiplication. Together with the characterization (1) of $\check{R}^{\Box_u,\Box_v}$, this implies that $(1 \otimes x)\check{R}^{K,\text{top}_v}(x^{-1} \otimes 1) = \check{R}^{K,\text{top}_v}$ for $x \in \langle g, h\rangle$ as desired. The fact that $g(K, \text{top}_v) \neq 0$ follows in generic from the explicit calculation (appendix). To prove (iii), we use (i) and an elementary braid manipulation to get

$$\begin{aligned}\check{R}^{L,\mathrm{top}_u}(1_L\otimes\pi)(\check{R}^{K,L}\otimes 1_{K^*}) &= (1_{K^*}\otimes\check{R}^{L,K})(\check{R}^{L,K^*}\otimes 1_K)(1_L\otimes\check{R}^{K,K^*})(\check{R}^{K,L}\otimes 1_{K^*})\\ &= (\pi\otimes 1_L)(1_K\otimes\check{R}^{L,K^*})\cdot f(K,L)\\ &\quad : V(K)\otimes V(L)\otimes V(K^*)\longrightarrow V(\mathrm{top}_u)\otimes V(L)\end{aligned}$$

where $\pi=\check{R}^{K,K^*}: V(K)\otimes V(K^*)\longrightarrow V(\mathrm{top}_u)$. Rewriting this in terms of matrix elements we obtain (16). Similarly we get (17) from the identity

$$\check{R}^{\mathrm{top}_u,L}(\pi\otimes 1_L)(1_K\otimes\check{R}^{L,K^*}) = (1_L\otimes\pi)(\check{R}^{K,L}\otimes 1_{K^*})\cdot f(L,K^*).$$

□

Corollary 1. Comparing (16) and (17), we have

$$\frac{f(K,L)}{g(L,\mathrm{top}_u)}=\frac{g(\mathrm{top}_u,L)}{f(L,K^*)}.$$

Corollary 2. Using (16) twice, we have

$$(\check{R}^{K,L})^{IJ}_{J'I'}=(\check{R}^{K^*,L^*})^{I'J'}_{JI}\,\frac{f(K,L)}{g(L,\mathrm{top}_u)}\,\frac{f(L,K^*)}{g(K^*,\mathrm{top}_v)}.$$

Remark 2. For $K=1^k_u$ write $K^{*'}:=1^{n-k}_{u+(k-n)\hbar}$. We can also define a pairing by using $\check{R}^{K^{*'},K}$,

$$\langle,\rangle' : V(K^{*'})\otimes V(K)\to V(\mathrm{top}_{u+(k-n)\hbar})\to\mathbb{C} \tag{19}$$

which is also non-degenarate for generic $\hbar$ so that we can identify $V(K)^*$ and $V(K^{*'})$.

4. Crossing symmetry in the face models

As in the vertex case we can similarly derive the crossing symmetry for face models. Put

$$\check{W}^{\square_{u_1}\square_{u_2}\cdots\square_{u_l},\square_v} := (\check{W}^{\square_{u_1},\square_v})^{1,2}(\check{W}^{\square_{u_2},\square_v})^{2,3}\cdots(\check{W}^{\square_{u_l},\square_v})^{l,l+1}$$

$$\begin{aligned}\check{W}^{\square_{u_1}\cdots\square_{u_k},\square_{v_1}\cdots\square_{v_l}} &:= (\check{W}^{\square_{u_1}\cdots\square_{u_k},\square_{v_l}})^{k\cdots k+l-1;k+l}\\ &\quad\cdots(\check{W}^{\square_{u_1}\cdots\square_{u_k},\square_{v_2}})^{2\cdots k+1;k+2}(\check{W}^{\square_{u_1}\cdots\square_{u_k};\square_{v_1}})^{1\cdots k;k+1}\\ &\quad : \mathcal{P}_{\square_{u_1}\cdots\square_{u_k}}\otimes\mathcal{P}_{\square_{v_1}\cdots\square_{v_l}}\to\mathcal{P}_{\square_{v_1}\cdots\square_{v_l}}\otimes\mathcal{P}_{\square_{u_1}\cdots\square_{u_k}}\end{aligned}$$

where the superscripts denote the components they act on. The fusion operator for the face model [2] associated with 1^k is given by

$$\begin{aligned}\Pi_{1^k} &:= (\check{W}^{\square_{u_1},\square_{u_2}})^{k-1;k}\cdots(\check{W}^{\square_{u_1}\square_{u_2}\cdots\square_{u_{k-2}},\square_{u_{k-1}}})^{2\cdots k-1;k}(\check{W}^{\square_{u_1}\square_{u_2}\cdots\square_{u_{k-1}},\square_{u_k}})^{1\cdots k-1;k}\\ &\quad : \mathcal{P}_{\square_u\cdots\square_{u+(k-1)\hbar}}\to\mathcal{P}_{\square_{u+(k-1)\hbar}\cdots\square_u}\end{aligned} \tag{20}$$

where the spectral parameters are specialized as before (10): $(u_1, \ldots, u_k) = (u, \ldots, u + (k-1)\hbar)$. We denote the image of Π_{1^k} in $\mathcal{P}_{\square_{u+(k-1)\hbar}\cdots\square_u}$ (respectively $\mathcal{P}^{\nu}_{\lambda\square_{u+(k-1)\hbar}\cdots\square_u}$) as $\mathcal{P}_{1^k_u}$ (respectively $\mathcal{P}^{\nu}_{\lambda 1^k_u}$), or $\mathcal{P}_K$ (respectively $\mathcal{P}^{\nu}_{\lambda K}$) using the shorthand from the previous section $K = 1^k_u$. We also write $\mathcal{P}^{\nu}_{\lambda KL} := \oplus_\mu \mathcal{P}^{\mu}_{\lambda K} \otimes \mathcal{P}^{\nu}_{\mu L}$, $\mathcal{P}_{KL} := \oplus_{\lambda\nu}\mathcal{P}^{\nu}_{\lambda KL}$. It turns out that for $K = 1^k_u$ and generic value of $\hbar$ the dimension of the space $\mathcal{P}^{\nu}_{\lambda K}$ is given by

$$\dim \mathcal{P}^{\nu}_{\lambda K} = |\{(j_1, \ldots, j_k); 1 \leqslant j_1 < \cdots < j_k \leqslant n, \bar{\epsilon_{j_1}} + \cdots + \bar{\epsilon_{j_k}} = \nu - \lambda\}|$$

which is equal to the multiplicity of the weight $\nu - \lambda$ of the $\mathrm{GL}(\mathbb{C}^n)$-module $\wedge^k(\mathbb{C}^n)$. In particular, for top $= 1^n$ we have $\dim \mathcal{P}^{\nu}_{\lambda \mathrm{top}_u} = \delta_{\lambda,\nu}$.

The fused weight for $K = 1^k_u, L = 1^l_v$ is defined by

$$\check{W}^{K,L} := \check{W}^{\square_{u+(k-1)\hbar}\cdots\square_u,\square_{v+(l-1)\hbar}\cdots\square_v}|_{\mathcal{P}_{KL}} : \mathcal{P}_{KL} \to \mathcal{P}_{LK}$$

and they satisfy

$$(\check{W}^{L,M} \otimes 1)(1 \otimes \check{W}^{K,M})(\check{W}^{K,L} \otimes 1) = (1 \otimes \check{W}^{K,L})(\check{W}^{K,M} \otimes 1)(1 \otimes \check{W}^{L,M}). \tag{21}$$

Fix a base $|\mathrm{top}_{\lambda,u}\rangle \in \mathcal{P}^{\lambda}_{\lambda \mathrm{top}_u}$ for each λ and u. We can define the pairing

$$\langle , \rangle : \mathcal{P}_K \otimes \mathcal{P}_{K^*} \to \mathcal{P}_{\mathrm{top}_u} \to \mathbb{C} \tag{22}$$

as the composition of $\check{W}^{K,K^*}$ and the identification map $|\mathrm{top}_{\lambda,u}\rangle \mapsto 1$. Suppose $\mathcal{P}^{\mu}_{\lambda K} \neq 0$. Then for generic $\hbar$ this pairing is non-degenarate between $\mathcal{P}^{\mu}_{\lambda K}$ and $\mathcal{P}^{\lambda}_{\mu K^*}$ so that we can identify $(\mathcal{P}^{\mu}_{\lambda K})^*$ and $\mathcal{P}^{\lambda}_{\mu K^*}$. Take basis $\{e^{\mu}_{\lambda a}\}_a$ of $\mathcal{P}^{\mu}_{\lambda K}$ and its dual basis (with respect to $\langle , \rangle$) $\{e^{*\lambda}_{\mu a}\}_a$ of $\mathcal{P}^{\lambda}_{\mu K^*}$ and define the matrix element of $\check{W}$ as follows.

$$\check{W}^{K,L}(e^{\mu}_{\lambda a} \otimes e^{\nu}_{\mu b}) = \sum_{b'\mu' a'} e^{\mu'}_{\lambda b'} \otimes e^{\nu}_{\mu' a'} \cdot \check{W}^{K,L}\begin{bmatrix} a & \mu & b \\ \lambda & & \nu \\ b' & \mu' & a' \end{bmatrix} \qquad (e^{\mu}_{\lambda a} \in \mathcal{P}^{\mu}_{\lambda K}, e^{\nu}_{\mu b} \in \mathcal{P}^{\nu}_{\mu L}).$$

Proposition 2. Let K, L are as in proposition 1 and let $f(K, L)$ be the scalar in (14).

(i) We have

$$\check{W}^{L,K}\check{W}^{K,L} = f(K, L) \cdot \mathrm{id}.$$

(ii) For each λ and μ there exists a scalar $G^{\mu}_{\lambda}(K, \mathrm{top}_v)$ (respectively $G^{\mu}_{\lambda}(\mathrm{top}_u, L)$) which is non-zero for generic u, v and satisfies

$$\check{W}^{K,\mathrm{top}_v}(b \otimes |\mathrm{top}_{\mu,v}\rangle) = |\mathrm{top}_{\lambda,v}\rangle \otimes b \cdot G^{\mu}_{\lambda}(K, \mathrm{top}_v)$$

$$(\text{respectively}\, \check{W}^{\mathrm{top}_u,L}(|\mathrm{top}_{\lambda,u}\rangle \otimes b) = b \otimes |\mathrm{top}_{\mu,u}\rangle G^{\mu}_{\lambda}(\mathrm{top}_u, L))$$

for any $b \in \mathcal{P}^{\mu}_{\lambda K}$ (respectively $\mathcal{P}^{\mu}_{\lambda L}$).

(iii) The crossing symmetry is given by

$$\check{W}^{K,L}\begin{bmatrix} a & \mu & b \\ \lambda & & \nu \\ b' & \mu' & a' \end{bmatrix} = \check{W}^{L,K^*}\begin{bmatrix} b & \nu & a' \\ \mu & & \mu' \\ a & \lambda & b' \end{bmatrix} \frac{f(K, L)}{G^{\mu'}_{\lambda}(L, \mathrm{top}_u)} \tag{23}$$

$$= \check{W}^{L,K^*}\begin{bmatrix} b & \nu & a' \\ \mu & & \mu' \\ a & \lambda & b' \end{bmatrix} \frac{G^{\mu'}_{\lambda}(\mathrm{top}_u, L)}{f(L, K^*)}. \tag{24}$$

Proof. Proof of (i) is similar to the vertex case. (ii) is trivial when $k = 1$ because the space $\mathcal{P}^{\mu}_{\lambda\Box_u}$ is, at most, one-dimensional, and then the general case follows. The fact that $G^{\mu}_{\lambda}(K, \mathrm{top}_v) \neq 0$ for generic $u - v$ follows from calculation (lemma 1 and the appendix). To prove (iii), as in the vertex case we use (i) and an elementary braid manipulation to obtain

$$\check{W}^{L,\mathrm{top}_u}(1 \otimes \Pi)(\check{W}^{K,L} \otimes 1) = (\Pi \otimes 1)(1 \otimes \check{W}^{L,K^*}) \cdot f(K, L)$$
$$: \mathcal{P}_K \otimes \mathcal{P}_L \otimes \mathcal{P}_{K^*} \longrightarrow \mathcal{P}_{\mathrm{top}_u} \otimes \mathcal{P}_L$$

Here $\Pi = \check{W}^{K,K^*}$. Rewriting this in terms of matrix elements we obtain (23). Similarly we get (24).

Corollary 3.

$$\frac{f(K, L)}{G^{\mu}_{\lambda}(L, \mathrm{top}_u)} = \frac{G^{\mu}_{\lambda}(\mathrm{top}_u, L)}{f(L, K^*)}.$$

Corollary 4.

$$\check{W}^{K,L}\begin{bmatrix} a & \mu & b \\ \lambda & & \nu \\ b' & \mu' & a' \end{bmatrix} = \check{W}^{K^*,L^*}\begin{bmatrix} a' & \mu' & b' \\ \nu & & \lambda \\ b & \mu & a \end{bmatrix} \frac{f(K, L)}{G^{\mu'}_{\lambda}(L, \mathrm{top}_u)} \frac{f(L, K^*)}{G^{\lambda}_{\mu}(K^*, \mathrm{top}_v)}. \tag{25}$$

Remark 3. As in the vertex case we can define a non-degenerate pairing

$$\langle , \rangle' : \mathcal{P}^{\mu}_{\lambda K^{*\prime}} \otimes \mathcal{P}^{\lambda}_{\mu K} \to \mathcal{P}^{\lambda}_{\lambda \mathrm{top}_{u+(k-n)\hbar}} \to \mathbb{C} \tag{26}$$

by using $\check{W}^{K^{*\prime},K}$ where $K^{*\prime}$ is the same as before (19).

5. The incoming intertwining vectors

Fusion of the intertwining vector was treated in [15] with the generalization of the vertex-face correspondence (8). Here we will review them in our formalism so that we can observe the algebraic struture directly. Let $K = 1^k_u, L = 1^l_v, \mathrm{top} = 1^n$ as before. Consider the operator

$$\phi^{\nu}_{\lambda\Box_{u_1}\Box_{u_2}\cdots\Box_{u_k}} := \oplus_{\mu_1,\mu_2\cdots,\mu_{k-1}} \phi^{\mu_1}_{\lambda\Box_{u_1}} \otimes \phi^{\mu_2}_{\mu_1\Box_{u_2}} \otimes \cdots \otimes \phi^{\nu}_{\mu_{k-1}\Box_{u_k}}$$
$$: \mathcal{P}^{\nu}_{\lambda\Box_{u_1}\Box_{u_2}\cdots\Box_{u_k}} = \oplus_{\mu_1,\mu_2\cdots,\mu_{k-1}} \mathcal{P}^{\mu_1}_{\lambda\Box_{u_1}} \otimes \mathcal{P}^{\mu_2}_{\mu_1\Box_{u_2}} \otimes \cdots \otimes \mathcal{P}^{\nu}_{\mu_{k-1}\Box_{u_k}}$$
$$\longrightarrow V(\Box_{u_1}) \otimes V(\Box_{u_2}) \otimes \cdots \otimes V(\Box_{u_k})$$

then from the intertwining property (8) we have

$$\pi_{1^k}\phi^{\nu}_{\lambda\Box_u\Box_{u+\hbar}\cdots\Box_{u+(k-1)\hbar}} = \phi^{\nu}_{\lambda\Box_{u+(k-1)\hbar}\cdots\Box_{u+\hbar}\Box_u}\Pi_{1^k}.$$

This implies that the image of the restriction $\phi^{\nu}_{\lambda K} := \phi^{\nu}_{\lambda\Box_{u+(k-1)\hbar}\cdots\Box_{u+\hbar}\Box_u}|_{\mathcal{P}^{\nu}_{\lambda K}}$ lies in $V(K)$:

$$\phi^{\nu}_{\lambda K} : \mathcal{P}^{\nu}_{\lambda K} \to V(K). \tag{27}$$

This is the fused intertwining vector. Let

$$\mathrm{pr}^{L,K}_{\lambda\mu_0\nu} : \oplus_{\mu'}\mathcal{P}^{\mu'}_{\lambda L}\otimes\mathcal{P}^{\nu}_{\mu' K}\to\mathcal{P}^{\mu_0}_{\lambda L}\otimes\mathcal{P}^{\nu}_{\mu_0 K}$$

denotes the projection and put

$$\check{W}^{K,L}\begin{bmatrix} & \mu & \\ \lambda & & \nu \\ & \mu' & \end{bmatrix} := \mathrm{pr}^{L,K}_{\lambda\mu'\nu}\cdot\check{W}^{K,L}|_{\mathcal{P}^{\mu}_{\lambda K}\otimes\mathcal{P}^{\nu}_{\mu L}} : \mathcal{P}^{\mu}_{\lambda K}\otimes\mathcal{P}^{\nu}_{\mu L}\to\mathcal{P}^{\mu'}_{\lambda L}\otimes\mathcal{P}^{\nu}_{\mu' K}.$$

Then the generalized vertex–face correspondence or the intertwining property [15] can be stated as follows

$$\check{R}^{K,L}\phi^{\mu}_{\lambda K}\otimes\phi^{\nu}_{\mu L}=\sum_{\mu'}\phi^{\mu'}_{\lambda L}\otimes\phi^{\nu}_{\mu' K}\check{W}^{K,L}\begin{bmatrix} & \mu & \\ \lambda & & \nu \\ & \mu' & \end{bmatrix}. \tag{28}$$

Here both sides are the operators $\mathcal{P}^{\mu}_{\lambda K}\otimes\mathcal{P}^{\nu}_{\mu L}\to V(L)\otimes V(K)$.

5.1. The incoming intertwining vectors

The fused intertwining vectors (27) may be called outgoing intertwining vectors because the space $V(K)$ appears there as the output of these quantities. In contrast to this, what we should like to call 'incoming' intertwining vectors are the quantities

$$\phi^{\mu L}_{\lambda} : V(L)\to\mathcal{P}^{\mu}_{\lambda L}$$

that satisfy

$$\phi^{\mu L}_{\lambda}\otimes\phi^{\nu K}_{\mu}\check{R}^{K,L}=\sum_{\mu'}\check{W}^{K,L}\begin{bmatrix} & \mu' & \\ \lambda & & \nu \\ & \mu & \end{bmatrix}\phi^{\mu' K}_{\lambda}\otimes\phi^{\nu L}_{\mu'} \tag{29}$$

and we are now in the position to construct them.

First we substitute k by $n-k$ and l by $n-l$ in (28),

$$\check{R}^{K^*,L^*}\phi^{\mu}_{\nu K^*}\otimes\phi^{\lambda}_{\mu L^*}=\sum_{\mu'}\phi^{\mu'}_{\nu L^*}\otimes\phi^{\lambda}_{\mu' K^*}\check{W}^{K^*,L^*}\begin{bmatrix} & \mu & \\ \nu & & \lambda \\ & \mu' & \end{bmatrix}$$

and take the matrix elements: write

$$\phi^{\nu}_{\lambda K}(e^{\nu}_{\lambda a})=\sum_{I}e^{I}(\phi^{\nu}_{\lambda K})_{I,a}\in V(K)$$

then we have

$$\sum_{IJ}(\check{R}^{K^*,L^*})^{IJ}_{J'I'}(\phi^{\mu}_{\nu K^*})_{Ia}(\phi^{\lambda}_{\mu L^*})_{Jb}=\sum_{\mu'a'b'}(\phi^{\mu'}_{\nu L^*})_{J'b'}(\phi^{\lambda}_{\mu' K^*})_{I'a'}\check{W}^{K^*L^*}\begin{bmatrix} a & \mu & b \\ \nu & & \lambda \\ b' & \mu' & a' \end{bmatrix}.$$

We use the crossing symmetries (18), (25) in corollaries 2, 4 to get

$$\sum_{IJ}(\check{R}^{K,L})^{I'J'}_{JI}\frac{g(L,\mathrm{top}_{\mu})}{f(K,L)}\frac{g(K^*,\mathrm{top}_{\nu})}{f(L,K^*)}(\phi^{\mu}_{\nu K^*})_{Ia}(\phi^{\lambda}_{\mu L^*})_{Jb}$$
$$=\sum_{\mu'a'b'}(\phi^{\mu'}_{\nu L^*})_{J'b'}(\phi^{\lambda}_{\mu' K^*})_{I'a'}\check{W}^{K,L}\begin{bmatrix} a' & \mu' & b' \\ \lambda & & \nu \\ b & \mu & a \end{bmatrix}\frac{G^{\mu}_{\lambda}(L,\mathrm{top}_{\mu})}{f(K,L)}\frac{G^{\lambda}_{\mu'}(K^*,\mathrm{top}_{\nu})}{f(L,K^*)}. \tag{30}$$

We need the following lemma.

Lemma 1. Let $\langle\phi^{\mu}_{\mu\mathrm{top}_v}\rangle \in \mathbb{C}$ be the coefficient in the formula

$$\phi^{\mu}_{\mu\mathrm{top}_v}|\mathrm{top}_{\mu,v}\rangle = |\mathrm{top}_v\rangle\langle\phi^{\mu}_{\mu\mathrm{top}_v}\rangle$$

where $|\mathrm{top}_{\mu,v}\rangle \in \mathcal{P}^{\mu}_{\mu\mathrm{top}_v}$ (respectively $|\mathrm{top}_v\rangle \in V(\mathrm{top}_v)$) denotes a fixed basis of the one-dimensional space. Then we have the formula

$$\frac{G^{\mu}_{\lambda}(K,\mathrm{top}_v)}{g(K,\mathrm{top}_v)} = \frac{\langle\phi^{\mu}_{\mu\mathrm{top}_v}\rangle}{\langle\phi^{\lambda}_{\lambda\mathrm{top}_v}\rangle}. \tag{31}$$

Proof. Recall the formula (28) for $l = n$,

$$\check{R}^{K,\mathrm{top}_v}\phi^{\mu}_{\lambda K} \otimes \phi^{\mu}_{\mu\mathrm{top}_v} = \sum_{\mu'} \phi^{\mu'}_{\lambda\mathrm{top}_v} \otimes \phi^{\mu}_{\mu' K}\check{W}^{K,\mathrm{top}_v}\begin{bmatrix} & \mu & \\ \lambda & & \mu \\ & \mu' & \end{bmatrix}.$$

Since $\dim\mathcal{P}^{\mu}_{\lambda\mathrm{top}_u} = \delta_{\lambda\mu}$, the summand on the right-hand side is zero unless $\mu' = \lambda$. Then the lemma follows as we take the matrix elements of both sides with using proposition 2(ii). □

Applying (31) to (30) we have

$$\begin{aligned}
&\sum_{IJ}(\check{R}^{K,L})^{I'J'}_{JI}\cdot(\phi^{\mu}_{vK^*})_{Ia}(\phi^{\lambda}_{\mu L^*})_{Jb}\\
&\quad= \sum_{\mu'a'b'}(\phi^{\mu'}_{vL^*})_{J'b'}(\phi^{\lambda}_{\mu'K^*})_{I'a'}\check{W}^{K,L}\begin{bmatrix} a' & \mu' & b' \\ \lambda & & v \\ b & \mu & a \end{bmatrix}\frac{G^{\mu}_{\lambda}(L,\mathrm{top}_u)}{g(L,\mathrm{top}_u)}\frac{G^{\lambda}_{\mu'}(K^*,\mathrm{top}_v)}{g(K^*,\mathrm{top}_v)}\\
&\quad= \sum_{\mu'a'b'}(\phi^{\mu'}_{vL^*})_{J'b'}(\phi^{\lambda}_{\mu'K^*})_{I'a'}\check{W}^{K,L}\begin{bmatrix} a' & \mu' & b' \\ \lambda & & v \\ b & \mu & a \end{bmatrix}\frac{\langle\phi^{\mu}_{\mu\mathrm{top}_u}\rangle}{\langle\phi^{\lambda}_{\lambda\mathrm{top}_u}\rangle}\frac{\langle\phi^{\lambda}_{\lambda\mathrm{top}_v}\rangle}{\langle\phi^{\mu'}_{\mu'\mathrm{top}_v}\rangle}.
\end{aligned}$$

Dividing both sides by $\langle\phi^{\mu}_{\mu\mathrm{top}_u}\rangle\langle\phi^{\lambda}_{\lambda\mathrm{top}_v}\rangle$ we get

$$\sum_{IJ}(\check{R}^{K,L})^{I'J'}_{JI}\frac{(\phi^{\mu}_{vK^*})_{Ia}}{\langle\phi^{\mu}_{\mu\mathrm{top}_u}\rangle}\frac{(\phi^{\lambda}_{\mu L^*})_{Jb}}{\langle\phi^{\lambda}_{\lambda\mathrm{top}_v}\rangle} = \sum_{\mu'a'b'}\frac{(\phi^{\mu'}_{vL^*})_{J'b'}}{\langle\phi^{\mu'}_{\mu'\mathrm{top}_v}\rangle}\frac{(\phi^{\lambda}_{\mu'K^*})_{I'a'}}{\langle\phi^{\lambda}_{\lambda\mathrm{top}_u}\rangle}\check{W}^{K,L}\begin{bmatrix} a' & \mu' & b' \\ \lambda & & v \\ b & \mu & a \end{bmatrix}.$$

Thus we obtained the desired incoming intertwining vectors (29).

Theorem 1. For each λ, v and $K = 1^k_u$ define the operator

$$\phi^{vK}_{\lambda} : V(K) \to \mathcal{P}^{v}_{\lambda K}$$

by

$$\phi^{vK}_{\lambda}(e^I) := \sum_a e^{v}_{\lambda a}\frac{(\phi^{\lambda}_{vK^*})_{Ia}}{\langle\phi^{\lambda}_{\lambda\mathrm{top}_u}\rangle}$$

for each basis element $e^I \in V(K)$, where $K^* := 1^{n-k}_{u+k\hbar}$ (12). Then they satisfy

$$\phi^{\mu L}_{\lambda} \otimes \phi^{vK}_{\mu}\check{R}^{K,L} = \sum_{\mu'}\check{W}^{K,L}\begin{bmatrix} & \mu' & \\ \lambda & & v \\ & \mu & \end{bmatrix}\phi^{\mu' K}_{\lambda} \otimes \phi^{vL}_{\mu'}$$

where both sides are the operators $V(K) \otimes V(L) \to \mathcal{P}^{\mu}_{\lambda L} \otimes \mathcal{P}^{v}_{\mu K}$.

Remark. In the above definition (32) we used the dual basis for $(V(K), V(K^*))$ with respect to the pairing (13) as well as those for $(\mathcal{P}^{\mu}_{\lambda K}, \mathcal{P}^{\lambda}_{\mu K^*})$ with respect to (22). In defining these pairings we fixed the basis elements in $V(\mathrm{top}_u)$ and $\mathcal{P}^{\lambda}_{\lambda\mathrm{top}_u}$ respectively, but it can be checked easily that the quantity (32) does not depend on the choice of these basis. □

By construction, the incoming vectors above and the outgoing ones obey the following duality relations [10].

Proposition 3. Assume that $\mathcal{P}^{\mu}_{\lambda K} \neq 0$. Then we have

$$\phi^{\mu K}_{\lambda} \phi^{\mu}_{\nu K} = \delta_{\lambda,\nu} \mathrm{id}_{\mathcal{P}^{\mu}_{\lambda K}} : \mathcal{P}^{\mu}_{\nu K} \to V(K) \to \mathcal{P}^{\mu}_{\lambda K} \tag{33}$$

$$\sum_{\lambda} \phi^{\mu}_{\lambda K} \phi^{\mu K}_{\lambda} = \mathrm{id}_{V(K)} : V(K) \to \mathcal{P}^{\mu}_{\lambda K} \to V(K). \tag{34}$$

Proof. From the definition of the fused intertwining vector and the generalized vertex–face correspondence (28) we have

$$\check{R}^{K,K^*} (\phi^{\mu}_{\nu K} \otimes \phi^{\lambda}_{\mu K^*}) = \phi^{\lambda}_{\nu\mathrm{top}_u} \check{W}^{K,K^*} : \mathcal{P}^{\mu}_{\nu K} \otimes \mathcal{P}^{\lambda}_{\mu K^*} \to V(\mathrm{top}_u).$$

Evaluating this identity with the dual basis we have

$$\sum_{I} (\phi^{\mu}_{\nu K})_{I,a} (\phi^{\lambda}_{\mu K^*})_{I,b} = \delta_{\nu,\lambda} \delta_{a,b} \langle \phi^{\lambda}_{\lambda\mathrm{top}_u} \rangle.$$

Here the assumption $\mathcal{P}^{\mu}_{\lambda K} \neq 0$ is necessary, otherwise both sides will be 0. Together with the definition (32), this implies (33) and (34). □

Remark 5. To get the incoming intertwining vectors as above, we can take the pairings $\langle\rangle'$ (19),(26) instead of $\langle\rangle$ (13),(22) in the very beginning of the story. Then the whole construction works in the same way, but the resulting intertwining vectors, say $\phi'^{\mu K}_{\lambda}$, satisfy the duality relations (33),(34) in a slightly different form: $\phi'^{\mu K}_{\lambda} \phi^{\nu}_{\lambda K} = \delta_{\mu,\nu} \mathrm{id}_{\mathcal{P}^{\nu}_{\lambda K}}$, $\sum_{\mu} \phi^{\mu}_{\lambda K} \phi'^{\mu K}_{\lambda} = \mathrm{id}_{V(K)}$. □

6. The L-operator

We define the vector space

$$\mathcal{V} := \Pi_{\mu \in h^*} \mathbb{C} \delta^{\mu} \tag{35}$$

with δ^{μ}, the 'delta function supported at $\mu \in h^*$', as its basis.

Theorem 2. For each $\lambda, \mu \in h^*$ put

$$\check{L}(K)^{\mu}_{\lambda} := \phi^{\mu}_{\lambda K} \phi^{\mu K}_{\lambda} : V(K) \to \mathcal{P}^{\mu}_{\lambda K} \to V(K)$$

for $K = 1^k_u$ and define the operator

$$\check{L}(K) : V(K) \otimes \mathcal{V} \to \mathcal{V} \otimes V(K)$$

by

$$\check{L}(K)(v\otimes\delta^{\mu}):=\sum_{\lambda}\delta^{\lambda}\otimes\check{L}(K)^{\mu}_{\lambda}(v)$$

for any $v\in V(K)$ and $\mu\in h^*$. Then this operator is well defined and satisfies the following.

$$(\check{L}(L)\otimes 1)(1\otimes\check{L}(K))(\check{R}^{K,L}\otimes 1)=(1\otimes\check{R}^{K,L})(\check{L}(K)\otimes 1)(1\otimes\check{L}(L))$$

where $K=1^k_u$, $L=1^l_v$ and both sides are operators

$$V(K)\otimes V(L)\otimes\mathcal{V}\to\mathcal{V}\otimes V(L)\otimes V(K).$$

In particular, putting $k=l=1$ the operator $\check{L}(\Box_v)$ gives an L-operator for Belavin's R-matrix $\check{R}^{\Box_u,\Box_v}$.

Proof. Remark that for each λ, $\check{L}(K)^{\lambda}_{\mu}=0$ for all but finite μ, which imply that the operator $\check{L}(K)$ is well defined.

Then by the intertwining properties (28) (29) we have the following for each λ and ν,

$$\begin{aligned}\sum_{\mu}\check{L}(L)^{\mu}_{\lambda}\otimes\check{L}(K)^{\nu}_{\mu}\check{R}^{K,L}&=\sum_{\mu}(\phi^{\mu}_{\lambda L}\phi^{\mu L}_{\lambda})\otimes(\phi^{\nu}_{\mu K}\phi^{\nu K}_{\mu})\check{R}^{K,L}\\&=\sum_{\mu}(\phi^{\mu}_{\lambda L}\otimes\phi^{\nu}_{\mu K})(\phi^{\mu L}_{\lambda}\otimes\phi^{\nu K}_{\mu})\check{R}^{K,L}\\&=\sum_{\mu}\phi^{\mu}_{\lambda L}\otimes\phi^{\nu}_{\mu K}\sum_{\mu'}\check{W}^{K,L}\begin{bmatrix}&\mu'&\\\lambda&&\nu\\&\mu&\end{bmatrix}\phi^{\mu' K}_{\lambda}\otimes\phi^{\nu L}_{\mu'}\\&=\check{R}^{K,L}\sum_{\mu'}(\phi^{\mu'}_{\lambda K}\otimes\phi^{\nu}_{\mu' L})(\phi^{\mu' K}_{\lambda}\otimes\phi^{\nu L}_{\mu'})\\&=\check{R}^{K,L}\sum_{\mu'}(\phi^{\mu'}_{\lambda K}\phi^{\mu' K}_{\lambda})\otimes(\phi^{\nu}_{\mu' L}\phi^{\nu L}_{\mu'})\\&=\check{R}^{K,L}\sum_{\mu'}\check{L}(K)^{\mu'}_{\lambda}\otimes\check{L}(L)^{\nu}_{\mu'}.\end{aligned}$$

This identity of operators $V(K)\otimes V(L)\to V(L)\otimes V(K)$ implies the assertion. □

Remark 6. Recall the definition $V(1^k_u):=\pi_{1^k}(V(\Box_u)\otimes\cdots\otimes V(\Box_{u+k\hbar}))$, where $V(\Box_u)$ is just a copy of $\mathbb{C}^n$ (section 3). This implies $V(1^k_{u+x})\cong V(1^k_u)$. Similarly $\mathcal{P}^{\mu}_{\lambda 1^k_{u+x}}\cong\mathcal{P}^{\mu}_{\lambda 1^k_u}$. So identify these spaces and denote them as $V(1^k)$, $\mathcal{P}^{\mu}_{\lambda 1^k}$ respectively. Then we have the operator

$$\check{L}(1^k_{u+x},1^k_u)^{\mu}_{\lambda}:=\phi^{\mu}_{\lambda 1^k_{u+x}}\phi^{\mu 1^k_u}_{\lambda}:V(1^k)\cong V(1^k_u)\to\mathcal{P}^{\mu}_{\lambda 1^k_u}\cong\mathcal{P}^{\mu}_{\lambda 1^k_{u+x}}\to V(1^k_{u+x})\cong V(1^k)$$

and we can define $\check{L}(1^k_{u+x},1^k_u)$ by

$$\check{L}(1^k_{u+x},1^k_u)(v\otimes\delta^{\mu}):=\sum_{\lambda}\delta^{\lambda}\otimes\check{L}(1^k_{u+x},1^k_u)^{\mu}_{\lambda}(\mathrm{v}).\tag{36}$$

Adapting the above identification of spaces we can say that the operators $\check{R}1^k_u$, 1^l_v and $\check{W}^{1^k_u,1^l_v}$ depend only on their deference $u-v$. Then we apply the above proof and get

$$(\check{L}(L_{+x},L)\otimes 1)(1\otimes\check{L}(K_{+x},K))(\check{R}^{K,L}\otimes 1)=(1\otimes\check{R}^{K,L})(1\otimes\check{L}(K_{+x},K))(\check{L}(L_{+x},L)\otimes 1)$$

where $K=1^k_u, K_{+x}=1^k_{u+x}, L=1^l_v, L_{+x}=1^l_{v+x}$.

6.1. Discussion

The L-operator given in this section defines a representation of the algebra of L-operators [20] on $\mathcal{V}$ (35). This is a large space but contains some series of sub/quotient representations.

First, let Th_l be the complex vector space spanned by theta functions of level $l \in \mathbb{Z}_{\geqslant 0}$ on the weight space $\boldsymbol{h}^*$ and $Th_l^{S_n}$ be its subspace consisting of the Weyl group (= the symmetric group S_n) invariants. The space $Th_l^{S_n}$ is spanned by the level l characters for the affine Lie algebra $A_{n-1}^{(1)}$ [24]. Assume $x = l\hbar$ in (36). Then we can restrict (the contragradient of) our representation to the space $Th_l^{S_n}$. This generalizes series (a) in Sklyanin's work [17]. We conjecture that this representation is equivalent to the fused representation on degree l symmetric tensors which was given by [14] and studied by [25]. Here equivalent means that after a suitable choice of bases the matrix element of our L-operator and the corresponding one for the fused L-operators are the same. When $l = 1$ this equivalence can be proved by examining the transformation rules like (1) of the matrix elements in the problem [16].

Second, letting $\hbar$ to be a rational number we get 'cyclic' representations as the quotient. This generalizes the series (b) in [17], and suggests the generalization of Kashiwara–Miwa's solution of the star-triangle equation.

The author would like to report these important aspects of our L-operator elsewhere.

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Appendix. Some formulae

Here we will give the formula for three important factors appearing in this article.

Formula 1. The factor f in (14)

$$f(\Box_u, \Box_v) = \frac{h(\hbar - u + v)h(\hbar + u - v)}{h(\hbar)^2} \tag{37}$$

$$f(1_u^k, 1_v^l) = \prod_{i=0}^{k-1}\prod_{j=0}^{l-1} f(\Box_{u+i\hbar}, \Box_{v+j\hbar}). \tag{38}$$

Proof. (37) is taken from [21]. Then (38) follows from the definition of the fused R-matrix. □

Formula 2. The factor g in (15) is given by

$$g(\Box_u, \mathrm{top}_v) = \frac{h(u - v + \hbar)\prod_{k=1}^{n-1} h(u - v - k\hbar)}{h(\hbar)^n} \tag{39}$$

$$g(1_u^k, \mathrm{top}_v) = \prod_{j=0}^{k-1} g(\Box_{u+j\hbar}, \mathrm{top}_v) \tag{40}$$

Proof. Note that we can write $g(\Box_u, \mathrm{top}_v) = g(u - v)$ since the R-matrix depends only on the difference of the spectral parameters.

Step 1. We shall examine the zeros of the function g. Recall the definition $\check{R}^{\Box_u,\mathrm{top}_v} = g(x) \times P$ (15) of the function g. We first observe that

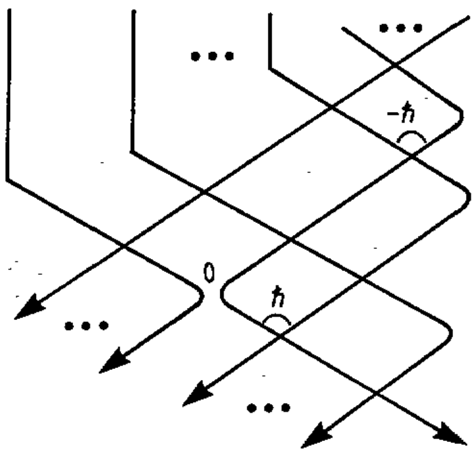
$$\check{R}^{\Box_u,\mathrm{top}_v}(1 \otimes \pi_{\mathrm{top}_v}) = 0 \tag{41}$$

if $u - v = -h$: In fact the left-hand side turns out to be the projector onto $V(1^{n+1}) \simeq \Lambda^{n+1}(\mathbb{C}^n) = 0$. Next, from (14) and (37) we have $\check{R}(x)\check{R}(-x) = 0$ for $x = \pm\hbar$. Then some braid manipulation shows that (41) also holds for $u - v = \hbar, 2\hbar, \ldots, (n-1)\hbar$.

$$\check{R}^{\Box_u,\mathrm{top}_v}(1 \otimes \pi_{\mathrm{top}_v})|_{u-v=k\hbar}$$

$$u = v + k\hbar \quad v \cdots v + (k-1)\hbar \quad v + k\hbar \cdots v + (n-1)\hbar$$

$=$



$$= 0 \qquad (1 \leqslant k \leqslant n-1).$$

Together with the periodicity of R, we conclude that

$$x = -\hbar; \hbar, 2\hbar, \ldots, (n-1)\hbar \bmod \mathbb{Z} + \tau\mathbb{Z}$$

gives rise to the zeroes of g.

Step 2. The transformation rule of $g(x)$ in x is easily deduced from that of R (1) and definition (15). We have

$$g(x+1) = g(x)(-1)^n$$

$$g(x+\tau) = g(x)(-1)^n \exp 2\pi \mathrm{i}[-n(\tfrac{1}{2}\tau + x) + (\tfrac{1}{2}n(n-1) - 1)\hbar].$$

Step 3. From steps 1, 2 and (6), a standard arguement in complex analysis shows that

$$g(x) = C \cdot h(x+\hbar) \prod_{j=1}^{n-1} h(x - j\hbar)$$

where C does not depend on x.

Step 4. To show $C = h(\hbar)^{-n}$, we shall investigate the transformation rules of $g(x) = g_\hbar(x)$ in $\hbar$. For that purpose we will consider the transformation rule of $R = R_\hbar(u)$ in $\hbar$. To do this, we observe the symmetrical roles of u and $\hbar$ in the Richey–Tracy formula (2), that is we have

$$R_\hbar(u)P = R_u(\hbar)\frac{H(u)}{H(\hbar)}$$

where

$$H(u) := \frac{\prod_{j=0}^{n-1}\theta^{(j)}(u)}{\prod_{j=1}^{n-1}\theta^{(j)}(0)} = h(u)(\frac{\eta(n\tau)}{\eta(\tau)})^3.$$

Put $\tilde{R}_\hbar(u) := h(\hbar)R_\hbar(u)$. Then the above observation and (1) implies the following.

$$\tilde{R}_{\hbar+1}(u) = (g\otimes 1)^{-1}\tilde{R}_\hbar(u)(1\otimes g)\times(-1)$$

$$\tilde{R}_{\hbar+\tau}(u) = (h\otimes 1)\tilde{R}_\hbar(u)(1\otimes h)^{-1}\times\left(-\exp 2\pi i\left(\hbar+\frac{u}{n}+\frac{\tau}{2}\right)\right)^{-1}.$$

We also remark here that, as a function in $\hbar$, $R_\hbar$ is not holomorphic but $\tilde{R}_\hbar(u)$ is. It follows that $\tilde{g}_\hbar(x) := h(\hbar)^n g_\hbar(x)$ is holomorphic in $\hbar$.

Step 5. From the previous step we can analyse the transformation rule of $\pi_{\rm top} = (\pi_{\rm top})_\hbar$ and $\check{R}^{\Box_{v+(n-1)\hbar}\cdots\Box_v,\Box_u} = (\check{R}^{\Box_{v+(n-1)\hbar}\cdots\Box_v,\Box_u})_\hbar$ and therefore that of $\tilde{g}_\hbar(x)$, in $\hbar$. The result is

$$\tilde{g}_{\hbar+1}(x) = \tilde{g}_\hbar(x)(-1)^{1+n(n-1)/2}$$

$$\tilde{g}_{\hbar+\tau}(x) = \tilde{g}_\hbar(x)(-1)^{1+n(n-1)/2}$$

$$\exp -2\pi i[(1-\tfrac{1}{2}n(n-1))x + (1+\tfrac{1}{6}n(n-1)(2n-1))(\hbar+\tfrac{1}{2}\tau)].$$

On the other hand, (42) gives that

$$\hbar = -x \bmod \mathbb{Z}+\tau\mathbb{Z}$$

$$= \frac{x+s+t\tau}{k} \bmod \mathbb{Z}+\tau\mathbb{Z} \qquad (0\leqslant s,t\leqslant k-1, 1\leqslant k\leqslant n-1)$$

giving zeros of $\tilde{g}_\hbar$ in $\hbar$. Then again some complex analysis shows that these properties determine the form of $\tilde{g}$: $\tilde{g}_\hbar(x) = \tilde{C}h(x+\hbar)\prod_{j=1}^{n-1}h(x-j\hbar)$, or

$$g_\hbar(x) = \tilde{C}\frac{h(x+\hbar)\prod_{j=1}^{n-1}h(x-j\hbar)}{h(\hbar)^n}$$

with $\tilde{C}$ being a constant in $\hbar$. The conclusion in step 2 implies $\tilde{C}$ is also a constant in x.

Step 7. Now we consider the limit of $\hbar\to 0$ to determine $\tilde{C}$: We have $\tilde{R}_\hbar(x)\to h(x)\cdot\mathrm{id}$, which implies $\tilde{g}_\hbar(x)\to h(x)^n$. Therefore $\tilde{C}=1$ and this completes the proof of (39).

Formula (40) follows from the definition of the fused R-matrix. □

Formula 3. With the appropriate choice of basis $|\mathrm{top}_{\lambda,u}\rangle \in \mathcal{P}^\lambda_{\lambda\mathrm{top}_u}$ and $|\mathrm{top}_u\rangle \in V(\mathrm{top}_u)$, we have

$$\langle\phi^\lambda_{\lambda\mathrm{top}_u}\rangle = (\mathrm{i}\eta(\tau))^{n-1-n(n-1)/2}h(u-(n-1)\hbar+\frac{n-1}{2})\prod_{1\leqslant j<k\leqslant n}h(n\hbar\lambda_{k,j}) \qquad (43)$$

where $\lambda_{k,j} = \langle\lambda+\rho,\bar{\epsilon}_k-\bar{\epsilon}_j\rangle$ and $\eta(\tau) := \exp\frac{1}{24}2\pi i\tau\prod_{m=1}^{\infty}(1-\exp 2\pi i m\tau)$ denotes the Dedekind eta function.

Proof. We recall the antisymmetry

$$\check{W}\begin{bmatrix} & \lambda+\bar{\epsilon}_i & \\ \lambda & -\hbar & \lambda+\bar{\epsilon}_j+\bar{\epsilon}_k \\ & \lambda+\bar{\epsilon}_j & \end{bmatrix} = -\check{W}\begin{bmatrix} & \lambda+\bar{\epsilon}_i & \\ \lambda & -\hbar & \lambda+\bar{\epsilon}_j+\bar{\epsilon}_k \\ & \lambda+\bar{\epsilon}_k & \end{bmatrix}$$

of the special value of the face weight which appears as a factor of the fusion operator Π_{1^m} (20). For the case $m = n$ it follows that any element in $\mathcal{P}^{\lambda}_{\lambda\mathrm{top}_u} \subset \mathcal{P}^{\lambda}_{\lambda\Box_{u+(n-1)\hbar}\cdots\Box_u}$ is of the form

$$\sum_{w\in S_n} \mathrm{sgn}(w) e_{\lambda}^{\lambda+\bar{\epsilon}_{w(1)}} \otimes e_{\lambda+\bar{\epsilon}_{w(1)}}^{\lambda+\bar{\epsilon}_{w(1)}+\bar{\epsilon}_{w(2)}} \otimes \cdots \otimes e_{\lambda+\bar{\epsilon}_{w(1)}+\cdots+\bar{\epsilon}_{w(n-1)}}^{\lambda+\bar{\epsilon}_{w(1)}+\cdots+\bar{\epsilon}_{w(n)}} \tag{44}$$

up to a scalar factor, where S_n stands for the symmetric group. To choose the basis $|\mathrm{top}_{\lambda,u}\rangle \in \mathcal{P}^{\lambda}_{\lambda\mathrm{top}_u}$ we shall fix the scalar to be 1 for all λ and u: $|\mathrm{top}_{\lambda,u}\rangle :=$ (44). Similarly we take

$$|\mathrm{top}_u\rangle := \sum_{w\in S_n} \mathrm{sgn}(w) e^{w(1)} \otimes \cdots \otimes e^{w(n)} \in V(\Box_{u+(n-1)\hbar}) \otimes \cdots \otimes V(\Box_u)$$

as the basis of $V(\mathrm{top}_u)$. We have $\langle e_1 \otimes \cdots \otimes e_n | \mathrm{top}_u\rangle = 1$, where $e_j \in V(\Box)^*$ is the dual basis for $\{e^j\} \subset V(\Box)$: $\langle e_j | e^k\rangle = \delta_{j,k}$. Using the standard bracket notation we have

$$\begin{aligned}\langle \phi^{\lambda}_{\lambda\mathrm{top}_u}\rangle &= \left\langle e_1 \otimes \cdots \otimes e_n \middle| \phi^{\lambda}_{\lambda\Box_{u+(n-1)\hbar}\cdots\Box_u} \middle| \sum_{w\in S_n} \mathrm{sgn}(w) e_{\lambda}^{\lambda+\bar{\epsilon}_{w(1)}} \otimes \cdots \otimes e_{\lambda+\bar{\epsilon}_{w(1)}+\cdots+\bar{\epsilon}_{w(n-1)}}^{\lambda+\bar{\epsilon}_{w(1)}+\cdots+\bar{\epsilon}_{w(n)}} \right\rangle \\ &= \sum_{w\in S_n} \mathrm{sgn}(w) (\phi^{\lambda+\bar{\epsilon}_{w(1)}}_{\lambda\Box_{u+(n-1)\hbar}})_1 \cdots (\phi^{\lambda+\bar{\epsilon}_{w(1)}+\cdots+\bar{\epsilon}_{w(n)}}_{\lambda+\bar{\epsilon}_{w(1)}+\cdots+\bar{\epsilon}_{w(n-1)}\Box_u})_n \\ &= \det[\theta^{(j)}(u-(n-1)\hbar - n\hbar\langle\lambda+\rho, \bar{\epsilon}_{w(k)}\rangle)]_{j,k=1,\ldots,n}.\end{aligned}$$

The Weyl–Kac denominator formula for $A^{(1)}_{n-1}$ reads as [24, 26]

$$\det[\theta^{(j)}(u_k)]_{j,k=1,\ldots,n} = h\left(\tfrac{1}{2}(n-1) + \sum_k u_k\right) \prod_{1\leqslant j<k\leqslant n} h(u_k - u_j)(\mathrm{i}\eta(\tau))^{n-1-n(n-1)/2}$$

and now the result follows. □

From the formula in this section and the relations in corollaries 1 and 3 and lemma 1, we can calculate the factors in the crossing symmetry in the vertex case (16), (17) as well as in the face case (23), (24). For example, we can write down the factor in (16) as follows [27],

$$\frac{f(\Box_u, \Box_v)}{g(\Box_v, \mathrm{top}_u)} = -\prod_{k=2}^{n-1} \frac{-h(\hbar)}{h(u-v+k\hbar)}. \tag{45}$$

Note added in proof. The author was introduced to the preprints [28, 29] after this work had been done, and also [27] during the revision. In [28] and [29] the incoming intertwining vectors for the vector representation ($\phi^{\mu\Box}_{\lambda}$ in this paper) is obtained by solving the duality relations (33) and (34) directly and a generalization of the Kashiwara–Miwa solution was studied. In [29], the crossing factor (16) and (45) was determined in a slightly different way from ours in the appendix. We used the transformation rule with respect to the parameter $\hbar$ and this is a basis-free arguement, while [27] uses a special matrix element.

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